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Explicit Guidance for Aeroassisted Orbital Plane Change

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Introduction

OVER the past several years, considerable research has been done on both coplanar and noncoplanar aeroassisted orbital transfer.[1] For noncoplanar orbital transfer, the guidance schemes based on the optimal trajectories have been of particular interest.[2] For re-entry lifting vehicles, it has been shown that there are two different kinds of control strategy for optimal plane change.[3,4] In the first control strategy, the vehicle initially pulls down to the denser atmosphere but, before the minimum altitude is reached, the vehicle pulls up to leave the atmosphere. In the second strategy, the vehicle initially pulls up to avoid entering too deeply into the denser atmosphere. Once the minimum altitude is reached, the pullup level decreases and the vehicle leaves the atmosphere. It may be argued that, for the first kind of maneuver, the vehicle has a slightly shallow entry angle, and the control is used to overcome the tendency of pulling up. For the second kind of maneuver, the vehicle enters the atmosphere with a slightly larger entry angle. Because there is a natural tendency to pull up, a slightly smaller control then is needed to pull up the vehicle. This overview suggests the problem of finding an optimal entry angle, so that the vehicle does not need to pull down or pull up for the given plane change. This note presents an explicit guidance law of a re-entry aeroassisted orbital vehicle. The vehicle enters the atmosphere with a prescribed entry angle and keeps the bank angle at 90 deg until exit. The entry angle is a function of the entry speed and the exit speed. The lift control in the atmosphere can be divided into two phases. In the first phase, the vehicle descends and keeps the normalized lift coefficient constant. During ascent, the normalized lift coefficient decreases proportionally to the increase in height. A comparison is made for the orbital plane change between the optimal aeroassisted trajectory and the trajectory realized by the explicit guidance. The proposed guidance scheme is simple to implement and produces nearly optimal plane change.

Dimensionless Equations of Motion

The motion of the re-entry vehicle over a spherical nonrotating planet is defined by the six variables, r (radius), θ (longitude), ϕ (latitude), V (velocity), γ (flight-path angle), and ψ (heading).[5]

Using a parabolic drag polar of the form, $C_D = C_{D_0} + KC_L^2$, we define the normalized lift coefficient $\lambda = C_L/C_L^*$, where C_L^* is the lift coefficient corresponding to the maximum lift-to-drag ratio E^*. The atmosphere is assumed to be at rest with respect to the planet and with a locally exponential property, $\rho = \rho_0 e^{-\beta(r-r_0)}$. The gravitational force field is given by the usual inverse-square force law. Without loss of generality, we can use the equatorial plane as the reference plane. We introduce the dimensionless variables $u = V^2/g_0 r_0$ and $h = (r - r_0)/r_0$ to represent speed and range, respectively, and define the dimensionless arc length as

$$s = \int_0^t \frac{V}{r}\cos\gamma \,\mathrm{d}t \tag{1}$$

to replace time as the independent variable. The coefficient B, which specifies the starting flight altitude, is given by

$$B = \frac{\rho_0 S C_L^* r_0}{2m} \tag{2}$$

where S is the reference area. The dimensionless equations of motion, with the bank angle $\sigma = 90$ deg, are

$$\frac{\mathrm{d}h}{\mathrm{d}s} = (1+h)\tan\gamma \tag{3a}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\cos\psi}{\cos\phi} \tag{3b}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \sin\psi \tag{3c}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}s} = \frac{B\lambda(1+h)e^{-h/\varepsilon}}{\cos^2\gamma} - \cos\psi\tan\phi \tag{3d}$$

$$\frac{\mathrm{d}u}{\mathrm{d}s} = -\frac{B(1+h)u(1+\lambda^2)e^{-h/\varepsilon}}{E^*\cos\gamma} - \frac{2}{1+h}\tan\gamma \tag{3e}$$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = 1 - \frac{1}{u(1+h)} \tag{3f}$$

Note that $\varepsilon = 1/\beta r_0$, which characterizes the atmosphere. In these dimensionless equations, the only physical characteristics to be specified are the vehicle's maximum lift-to-drag ratio and the coefficient B. In this problem, the choice of $B = 0.006$ specifies the initial entry altitude and is in the region suggested by Vinh et al.[6]

In the problem of optimal plane change, the initial values of states are

$$\begin{gathered} h_0 = 0, \quad \theta_0 = 0, \quad \phi_0 = 0, \quad \psi_0 = 0 \\ u_0 = \text{given}, \quad \gamma_0 = \text{free} \end{gathered} \tag{4a}$$

and the final values are

$$\begin{gathered} h_f = \text{free}, \quad \theta_f = \text{free}, \quad \phi_f = \text{free} \\ \psi_f = \text{free}, \quad u_f = \text{given}, \quad \gamma_f = \text{free} \end{gathered} \tag{4b}$$

Note that in this problem the lift is the only control.

Variational Formulation

Because the plane change I_f is going to be maximized, we have the following cost function J to be minimized:

$$J = \cos I_f = \cos\phi_f \cos\psi_f \tag{5}$$

Using the maximum principle, we introduce the adjoint variables p_x to form the Hamiltonian

$$\begin{aligned} H = {} & p_h(1+h)\tan\gamma + p_\theta \frac{\cos\psi}{\cos\phi} + p_\phi \sin\psi \\ & + p_\psi\left[\frac{B\lambda(1+h)e^{-h/\varepsilon}}{\cos^2\gamma} - \cos\psi\tan\phi\right] \\ & - p_u\left[\frac{B(1+h)u(1+\lambda^2)e^{-h/\varepsilon}}{E^*\cos\gamma} + \frac{2}{1+h}\tan\gamma\right] \\ & + p_\gamma\left[1 - \frac{1}{u(1+h)}\right] \end{aligned} \tag{6}$$

Presented as Paper 94-3527 at the AIAA Atmospheric Flight Mechanics Conference, Scottsdale, AZ, Aug. 1–3, 1994; received June 15, 1995; revision received June 14, 1996; accepted for publication June 15, 1996.

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By Pontryagin's maximum principle, the lift control can be derived from $\partial H/\partial \lambda = 0$, and it gives

$$\lambda = \frac{E^* p_\psi}{2 p_u u \cos\gamma} \tag{7}$$

Along the optimal trajectory, the adjoint variables associated with the state variables satisfy the adjoint equations

$$\frac{\mathrm{d}p_x}{\mathrm{d}s} = -\frac{\partial H}{\partial x} \tag{8}$$

The transversality conditions are

$$\begin{aligned} p_{\phi_f} &= -\frac{\partial J}{\partial \phi_f} = \sin\phi_f \cos\psi_f \\ p_{\psi_f} &= -\frac{\partial J}{\partial \psi_f} = \cos\phi_f \sin\psi_f \end{aligned} \tag{9}$$

It is known that the problem has the following integrals[1]:

$$H = c_0, \qquad p_\theta = c_1, \qquad p_\phi = c_2 \sin\theta - c_3 \cos\theta$$

and

$$p_\psi = c_1 \sin\phi + (c_2 \cos\theta + c_3 \sin\theta)\cos\phi \tag{10}$$

where the c_i are constants of integration. In this problem of optimal plane change, the final arc length s_f and final longitude θ_f are not prescribed. Hence, by the transversality conditions, we have

$$c_0 = 0, \qquad c_1 = 0 \tag{11}$$

At the final time

$$\frac{p_{\psi_f}}{p_{\phi_f}} = \frac{\tan\psi_f}{\tan\phi_f} = -\tan(\eta + \theta_f)\cos\phi_f \tag{12}$$

Explicitly, the transversality condition to be satisfied at the final time is

$$\tan\psi_f + \sin\phi_f \tan(\eta + \theta_f) = 0 \tag{13}$$

where $\eta = \tan^{-1}(c_2/c_3)$.

With the four integrals, only two of the remaining adjoint equations need to be integrated. The integration requires guessing two initial values, which, together with c_2 and c_3, constitute a four-parameter problem. By normalizing the adjoint variables, we obtain a three-parameter problem. Still, it is difficult to guess these parameters. The difficulty can be slightly alleviated by using the control λ, as given in Eq. (7), to replace the adjoint variables. By taking the derivatives of the equation, using the equations for the adjoint variables, Eq. (8), we directly obtain the equation for the control:

$$\begin{aligned} \frac{\mathrm{d}\lambda}{\mathrm{d}s} &= -\lambda \sin\psi \tan\phi + \lambda \tan\gamma \left[\frac{1}{u(1+h)} + 1\right] \\ &\quad - G\frac{\cos\psi}{\cos\gamma} + \frac{2\lambda K}{E^* u(1+h)} \end{aligned} \tag{14}$$

where K, F, and G are defined as the ratios

$$K = \frac{E^* p_\gamma}{2 u p_u}, \quad F = \frac{E^* p_h (1+h)}{2 u p_u}, \quad \text{and} \quad G = \frac{E^* p_\phi}{2 u p_u} \tag{15}$$

The equations for K, F, and G also can be obtained easily as

$$\begin{aligned} \frac{\mathrm{d}K}{\mathrm{d}s} &= \frac{2K}{E^* u(1+h)}(K + E^* \tan\gamma) + \frac{1}{\cos^2\gamma} \\ &\quad \times \left[\frac{B(1+h)e^{-h/\varepsilon}}{2}\sin\gamma(1-\lambda^2) + \frac{E^*}{u(1+h)} - F\right] \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}s} &= \frac{B(1+h)e^{-h/\varepsilon}}{2\cos\gamma}\left[1 - \frac{1+h}{\varepsilon}\right](1-\lambda^2) \\ &\quad + \frac{2F - E^*}{E^* u(1+h)}(K + E^* \tan\gamma) \end{aligned} \tag{17}$$

and

$$\frac{\mathrm{d}G}{\mathrm{d}s} = \frac{2G}{E^* u(1+h)}(K + E^* \tan\gamma) + \lambda\frac{\cos\psi\cos\gamma}{\cos^2\phi} \tag{18}$$

In summary, the six state equations (1) and the four equations directly providing the control λ and the accessory variables K, F, and G lead to 10 differential equations. Because the initial flight-path angle is free, the transversality condition implies that the K variable is initially zero. Using the boundary conditions given in Eq. (4), their integration requires guessing three initial parameters, λ_0, γ_0, and G_0, at the initial time, whereas F_0 can be computed from the Hamiltonian:

$$\begin{aligned} H = \frac{2up_u}{E^*}\bigg\{ & F\tan\gamma + G\sin\psi - \frac{E^*\tan\gamma}{u(1+h)} - \lambda\cos\gamma\cos\psi\tan\phi \\ & + K\left[1 - \frac{1}{u(1+h)}\right] - \frac{B(1+h)e^{-h/\varepsilon}}{2\cos\gamma}(1-\lambda^2)\bigg\} = 0 \end{aligned} \tag{19}$$

With the guessed constants and the initial values of the state variable, the integration stops when u_f reaches its prescribed values. The conditions of $h_f = 0$, $K_f = 0$, and Eq. (13) are then checked.

Let the vehicle having the characteristics of $E^* = 1.5$ enter the Earth's atmosphere with $\varepsilon = 1/900$ at the initial height $B = 0.006$. For the case of the entry speed equal to 1.9 and the exit speed equal to 1.0, the initial values of lift coefficient, flight-path angle, and auxiliary constant are $\lambda_0 = 1.046936$, $\gamma_0 = -4.5705$ deg, and $G_0 = -0.1945382$. The corresponding plane change is 27.18 deg. We have almost the same values of plane change as the optimal solution of Hanson.[3] It is found that for the cases in which u_0 varies from 1.1 to 1.9, $u_f = 1.0$, and σ is maintained at 90 deg, the lift controls have values slightly greater than one and remain nearly constant during the descent and are decreased linearly with increasing height during ascent. The observations give a clue to the proposed guidance scheme, which depends on the states of the aeroassisted vehicle only.

Explicit Guidance Law

With the observations from the preceding section, an explicit guidance scheme of the normalized lift control is proposed as

$$\lambda = \lambda_0, \quad \text{for} \quad \dot{h} < 0 \qquad \text{and} \qquad \dot{\lambda} = -k\dot{h}, \quad \text{for} \quad \dot{h} \geq 0 \tag{20}$$

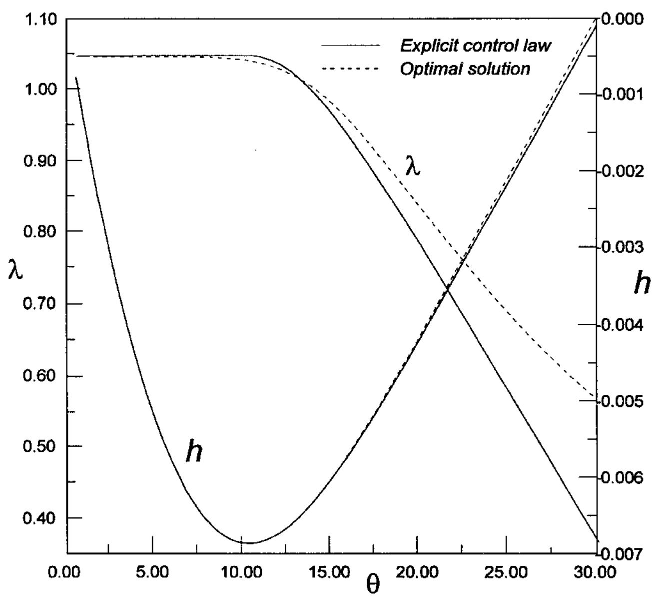

Fig. 1 Comparison of lift controls and trajectories.

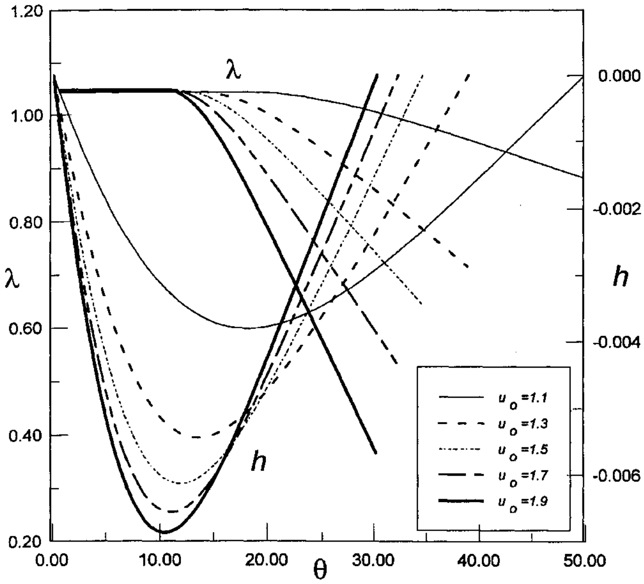

Fig. 2 Trajectories and associated lift controls.

In Eq. (20), λ_0 is the initial normalized lift coefficient from the optimal solution, and k is a constant to be adjusted to meet the final conditions (u_f = given, and $h_f = 0$). For the case of entry speed $u_0 = 1.9$ and exit speed $u_f = 1.0$, with the guidance scheme, using the value of λ_0 from the preceding section, the value of k is then adjusted to 99 to meet the final condition. Using this explicit guidance, the plane change is 27.159 deg, compared to the optimal value of 27.18 deg. Figure 1 shows comparisons of the lift controls and the trajectories for the optimal solution and the explicit guidance solution. The trajectories are almost the same, whereas the controls are slightly different. Several cases of different initial velocities are then studied. Figure 2 is a plot of the lift coefficients and the trajectories. Compared to the optimal solution of $\sigma = 90$ deg, the values of plane change are very good.

Conclusions

A simplified procedure for obtaining a suboptimal solution to the re-entry aeroassisted plane change is invented and an implementable guidance scheme for the lift control of the aeroassisted orbital plane change is proposed. The procedure reduces the number of variables to be guessed from three to two (the initial normalized lift coefficient and the proportional constant). The lifting vehicle enters the atmosphere with the optimal entry angle. The bank angle is maintained at 90 deg, and the lift control follows the simple rule during the atmospheric fly-through phase. The guidance scheme is quite simple, and the resulting trajectories are close to the optimal solutions. The plane change gained is also close to the optimal value.

Acknowledgment

This work was sponsored by the National Science Council, Republic of China, under Contract NSC84-2612-E-032-002.

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Determination of Equilibrium Points for an Aircraft Dynamic Model

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Nomenclature

A	$= b/c$
b, c	= wing span, m, and mean aerodynamic chord, m
C_T	= thrust coefficient, $T/0.5\,\rho V^2 S$
C_W	= weight coefficient, mg/0.5 $\rho V^2 S$
C_l, C_m, C_n	= aerodynamic-moment coefficients in body axes
C_x, C_y, C_z	= aerodynamic-force coefficients in body axes
g	= acceleration of gravity, m/s^2
H	= engine angular momentum, kg m^2/s
h	$= 4H/\rho S b^2 V_e$
I_x, I_y, I_z, I_{xz}	= moments and product of inertia; I_x, I_z, I_{xz} are dimensionless with respect to $\rho S(b/2)^3$ and I_y with respect to $\rho S(c/2)^3$
M	= Mach number
m	= mass, kg
p, q, r	= angular velocity components in body axes; q is dimensionless with respect to $2V_e/c$, p and r with respect to $2V_e/b$
S	= wing surface, m^2
T	= thrust, N
V	= flight speed, m/s
V_e	= equilibrium and reference velocity, m/s
v	= flight speed, dimensionless with respect to V_e
α, β	= aerodynamic angles
$\delta_a, \delta_e, \delta_r, \delta_l$	= aileron, elevator, rudder, and leading-edge flap angles, respectively
μ	= mass density, $2m/\rho Sc$
ρ	= air density, kg/m^3
ϕ, ϑ	= roll and pitch angles, respectively
ω	= angular velocity vector $(p, q, r)^T$

Introduction

A TECHNIQUE for the determination of the equilibrium points of the aircraft equations of motion is presented. The goal is, in particular, a method for the evaluation of the starting points for the continuation algorithms that are used in bifurcation analyses of the nonlinear dynamics of high-performance aircraft to trace the steady states of the system as certain parameters, called continuation parameters, are varied. In the application of bifurcation theory, the failure to determine as many starting points as possible, for an initial choice of the set of control variables, leads to the neglect of certain equilibrium branches, so that some behaviors of the dynamic system may not be revealed. This problem is far from being trivial because the number of equilibrium points is not known a priori and their determination involves the solution of a set of nonlinear, coupled algebraic equations.

In this respect, several theoretical and numerical techniques have been developed, stemming from semianalytical methods[1] to optimization algorithms.[2] In the latter approach a parameter search

Received July 10, 1995; revision received April 12, 1996; accepted for publication June 15, 1996.

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